An Answer to Certain Questions asked in the "Bulletin Astronomique," tome xiii, Mai 1896, *on Time Measurement.* By E. J. Stone, M.A., F.R.S., Radcliffe Observer.

In the *Bulletin Astronomique*, tome xiii, Mai 1896, there is a short review of my paper on the Corrections of the Epoch and Mean Motion required by Hansen's Lunar Tables.

The writer makes no remarks whatever on the accuracy with which the observations of the Moon 1750–1892 are represented by Hansen's Lunar Tables with the corrected epoch and mean motion; and I presume, therefore, that he is satisfied upon that point. See *Monthly Notices*, vol. lv. No. 2, p. 58.

But, with the exception of a slight ambiguity in the definition of the physical meaning of the linear quantity, a, which satisfies the identity

$$f = n^2 a^3,$$

a concise, clear, and accurate statement of the views upon which my work is based is given; and some questions are asked and some formulæ brought forward which deserve careful consideration.

The notation adopted is that, for a selected epoch, n represents the Earth's mean motion in heliocentric longitude in the adopted unit of time measured from the mean equinox at epoch; ω, the constant angular velocity of the Earth about its axis in the unit of time; f, the accelerating effects of the Sun's mass taken as the unit of mass in the unit of time at the unit of length; a, a linear constant which satisfies the condition

$$\left(\frac{f}{n^2}\right)^{\frac{1}{3}};$$

p and q, the mean regressions in longitude and right ascension respectively of the true equinox from the mean equinox at epoch:

$$\mathrm{N} = n + p : \Omega = \omega + q.$$

In order that these quantities may be combined in our mathematical investigations, it is, of course, absolutely necessary that n, ω, p, q, and f should be referred to the same unit of time, and f and a expressed in terms of the same unit of length, and that the same units should be employed for the expression of all the time quantities and linear quantities which appear in our formulæ.

The *exact* numerical values of the physical quantities represented by n, ω, p, q, f, and a will vary, according to well-known laws, with any changes which we may introduce into the units; and in tracing the effects of errors made we must proceed from exact equations to their approximate forms.

But if the units are not changed by us they cannot change.

The writer asks—

"Qu'est-on en droit de conclure lorsqu'on arrive, par une nouvelle discussion des observations, à une nouvelle valeur de $\frac{\omega}{n}$, l'invariabilité de la rotation de la Terre et de la constante de l'attraction étant admises ? Indépendamment de tout choix d'unités, on a

$$\frac{\omega}{n} = \mathrm{K}, \qquad f = n^2 a^3, \qquad \text{puis } \frac{\omega}{n'} = \mathrm{K}', \qquad f = n'^2 a'^3,$$

d'où

$$\left(\frac{a'}{a}\right)^3 = \left(\frac{n}{n'}\right)^2 = \left(\frac{\mathrm{K}'}{\mathrm{K}}\right)^2;$$

on ne peut conserver la même unité de longueur si le demi-grand axe calculé est toujours pris comme unité ; il y a contradiction et discontinuité. Il est clair qu'on ne peut espérer assurer la continuité absolue des unités de temps et de longueur alors qu'elles sont empruntées, pour une certaine part, au mouvement de la Terre autour du Soleil.

"Pour suivre M. Stone dans son raisonnement, nous voyons bien que dans une première discussion $\frac{\omega}{n}$ doit être, pour une valeur adoptée de l'unité de longueur, indépendant de l'unité de temps ; la même chose aura lieu pour les nombres provenant d'une seconde discussion ; mais quel droit a-t-on de conclure *a priori* à l'identité des deux valeurs de $\frac{\omega}{n}$ quelle que soit l'unité de longueur ?

"La substitution, en 1864, des Tables de Le Verrier à celles de Carlini a introduit une discontinuité dans l'unité de temps. Quelle a été, pour M. Stone, la modification de l'unité de longueur, et de quelle manière assure-t-il la continuité des mesures de longueur ?"

The differences between the angles n', or mean motion in geocentric longitude of the Sun from true equinox minus mean precessional motion, thus found from a discussion of solar observations, and the angle n adopted to fix the scale on which the accelerating effects of the different masses are measured, will be due to one of the following causes or their combination :—

Firstly. The difference may be due to errors in the adopted value of the precession constant, p, employed to pass from the mean motion in longitude measured from the true equinox to that from the mean equinox. The adopted value of the precessional constant should in this case be altered ; and it is clear that we should not change the angle n adopted for the Earth's mean motion in heliocentric longitude, measured from the equinox at epoch in the unit of time, from n to n', and the accelerating

effects of the Sun's mass in the unit of time from n^2 or n^2a^3 to n'^2 or n'^2a^3, because n' thus found differs from n.

Secondly. The difference may be due to errors in the observations. In such a case it is also clear that we should not change the angle adopted for the Earth's mean motion in heliocentric longitude in the unit of time from n to n', and the accelerating effects of the Sun's mass in the unit of time from n^2 or n^2a^3 to n'^2 or n'^2a^3, but should wait for a longer series of solar observations.

Thirdly. The difference may be due to the neglect in our theoretical expression of the Earth's longitude of long inequalities of the form

$$\mathrm{P} \sin (ct + d),$$

where P, c, and d are constants.

In this case, also, we ought not to change the angle adopted for the Earth's mean motion in heliocentric longitude from n to n', and the accelerating effects of the Sun's mass from n^2 or n^2a^3 to n'^2 or n'^2a^3; but the theory should be revised and the long inequalities computed on the scale fixed by the adopted angle n; and the effects should be duly allowed for in the discussion of the solar observations.

Fourthly. The difference between n and n' may be due to errors in our methods of finding the variable t from observation, or in referring the tabular positions of the centre of gravity of the Sun to the meridians of the observers. In this case, if t' is the value thus found, instead of t we should have, if the effects of the other sources of error were practically insensible,

$$(n' + p)t' = (n + p)t.$$

In such a case, as the variable t' is not the required variable t, we should, instead of altering n to n', correct the variable t'. It would not be of much practical importance whether we considered $n' - n$ as a correction on n, or $t - t'$ as a correction on t', if the disturbing effects of the planets were insensible; but as such is not the case, and these effects have to be estimated on the scale adopted in fixing the value of f, the use of t' for t will injuriously affect all our results proportionately to the magnitude of the factors of t, and corrections for the differences between t and t' must be applied in order to secure accuracy in the results. But perhaps it may be considered that no such error as the use of t' for t has been proved to have been made by astronomers. There is, however, no difficulty in proving that such errors are made in practice. But these proofs all turn upon the fundamental assumptions that *n represents the Earth's mean motion in longitude in the unit of time, and that we replace f by n^2a^3 in our mathematical work.*

In this case, if the physical quantities are denoted by n, ω, p, q, and f when referred to the unit of time fixed by the condition

$$f = n^2a^3,$$

and by $n_\circ$, $\omega_\circ$, $p_\circ$, $q_\circ$, and $f_\circ$ when referred to a "unit of time" fixed by the condition that $f_\circ = n_\circ^2 a^3$, where $n_\circ$ is such that

$$\omega_\circ + q_\circ = 365{\cdot}25 \times 2\pi + n_\circ + p_\circ,$$

or

$$\Omega_\circ = 365{\cdot}25 \times 2\pi + \mathrm{N}_\circ,$$

we shall have, with *exact* quantities,

$$\frac{\omega_\circ}{\omega} = \frac{n_\circ}{n} = \frac{q_\circ}{q} = \frac{p_\circ}{p},$$

$$\therefore\ \omega + q = 365{\cdot}25 \times 2\pi \times \frac{n}{n_\circ} + n + p,$$

$$\text{or } \Omega = 365{\cdot}25 \times 2\pi \times \frac{n}{n_\circ} + \mathrm{N},$$

$$\text{or } \Omega = 365{\cdot}25 \times 2\pi + \mathrm{N} + 365{\cdot}25 \times 2\pi \cdot \frac{n - n_\circ}{n_\circ}.$$

These formulæ also follow directly from the difference between the exact ratio of the mean sidereal day to the mean tropical year at epoch and that assumed by astronomers in practical work.

But astronomers in practical astronomical work erroneously replace

$$\Omega = 365{\cdot}25 \times 2\pi \times \frac{n}{n_\circ} + \mathrm{N},$$

$$\text{by } 365{\cdot}25 \times 2\pi + \mathrm{N},$$

although different values of n are adopted for the same epoch. And, as a consequence of the error thus made, when referring the positions of the centres of gravity of the Sun, Moon, and planets to the meridians of the observers, and in finding the time, t, from observation, they have taken out the *tabular* R.A. of the meridian, expressed in circular measure, for the time, t, in error by

$$-2\pi \cdot 365{\cdot}25 \cdot \frac{n_\circ - n}{n_\circ} \cdot t;$$

and all the terms in our theoretical expressions which contain t as a factor are thus proportionately affected with errors due to the neglected term.

It is true that the exact value of $n_\circ$, and therefore of

$$\frac{n_\circ - n}{n_\circ},$$

for an assigned value of n, and a definite epoch, is at present unknown; but if we include in the expressions for t, as directly found from observation, the effects of the neglected terms, we have the means of finding $n_\circ$ by continuous approximation from

the observations. And, although $n_0 - n$ is at present unknown, we know the difference in the errors made with two different adopted values n and n', viz. $n' - n$; and I have shown that the effects of the neglected terms due to the difference $n' - n$ for the values adopted for the Sun's mean motion in longitude in the unit of time in the solar tables which were employed, before and after 1864, to bring up the stellar places from one epoch to another, to reduce the geocentric places to heliocentric places, to refer the positions of the centres of gravity to the meridians of the observers, and to find the time, t, are sufficient to perfectly account for the increase in the errors of Hansen's Lunar Tables in longitude, which, as a matter of fact, set in *per saltum* about 1864, and has continued since, until it now amounts to about 22″.

I hope that I have here given a clear and distinct answer to the question, "What conclusion ought to be drawn when a new value n' of n results from a new discussion of solar observations?"

The difference between n and n' will be due to the causes which I have indicated, and in none of these cases is there any necessity for changing the numerical value of the accelerating effects of the Sun's mass at the unit of length in the unit of time; nor is there any means of evading the consequences if we make the change.

I now proceed to show why the writer of the review is led to physical inconsistencies by changing n into n'.

If the conditional equation

$$f = n^2 a^3$$

is true when any unit of time is adopted to measure the Sun's mean motion in longitude, which is here denoted by n, and any unit of length to measure the linear quantity, denoted by a, it is certainly true for any units of length and of time which we can adopt.

For f, the measure of the accelerating effect of the Sun's mass—

$$\propto \text{ as } \frac{(\text{the unit of time})^2}{(\text{the unit of length})^3},$$

and

$$n^2 \propto (\text{the unit of time})^2,$$

and

$$a^3 \propto \frac{1}{(\text{the unit of length})^3}.$$

The unit of time in terms of which f, n, ω are expressed is a matter of choice on the part of the mathematician, but *if n does express the Sun's mean motion in geocentric longitude in the unit of time* we certainly fix the unit of time when we assign a definite value to n, and the same unit of time must be adopted for the expression of f and ω. And, with regard to the unit of

length, we can, if we please, adopt the linear quantity, which satisfies the condition of being equal to

$$\left(\frac{f}{n^2}\right)^{\frac{1}{3}}$$

as the unit of length; and, with this unit, we have the accelerating effects of the Sun's mass accurately measured by n^2. *If we adopt these units in our mathematical work we clear f from fallible errors of determination, and it is a constant throughout our work*; but we have to express all the time quantities and the linear quantities in terms of the units of length and of time thus deliberately selected, and we must accept any practical inconveniences which may result from the choice of units thus made.

But we need not select this particular unit of length. We can select a, the linear constant introduced into the primary differential equations of the Sun's geocentric motion, or the semi-axis major of the undisturbed orbit as our unit of length. In this case we shall have, if ν_3 denotes the ratio of the mass of the Earth to that of the Sun,

$$f = n^2a^3 = \frac{n^2a^3}{1+\nu_3}\left(1+\frac{\delta a}{a}\right)^3 = \frac{n^2}{1+\nu_3}(1+\delta a)^3$$

when a is taken as the unit of length.

The correction δa will be a function of the disturbing masses, and must be found in terms of the adopted unit of length from the mathematical investigations. This is the method which Le Verrier followed, and it is clear that any change which we may make in n will change the unit of time in terms of which

$$\frac{n^2}{1+\nu_3} \cdot (1+\delta a)^3$$

will measure the accelerating effect of the Sun's mass, but not the unit of length; and that the unit of length is different when we adopt

$$n^2a^3, \quad n^2, \quad \frac{n^2}{1+\nu_3}(1+\delta a)^3$$

for the accelerating effects of the Sun's mass at the unit of length in the unit of time; and *neither the unit of length nor the unit of time can change unless we change them.*

Now the writer is directly led to physical inconsistencies and discontinuities, because, after adopting values of n, ω, and f, referred to a definite unit of time fixed by the condition that the assigned angle n shall be the Sun's mean motion in geocentric longitude in the unit of time, he assumes that he can adopt for the mean motion of the Sun in geocentric longitude in the unit of time an angle n' instead of n, and can yet assume that the values of ω and f are not proportionately changed. The conditions thus introduced are physically impossible relations between ω, f, and n'.

The results thus arrived at agree identically with those which I have obtained. I have shown that, unless we admit the possibility of discontinuities in our unit of time, we must, when we change n to n', change ω to ω', and f to f', such that

$$\frac{n}{n'}=\frac{\omega}{\omega'}=\frac{\sqrt{f}}{\sqrt{f'}}.$$

Similar remarks refer to the unit of length.

The linear quantity a cannot be changed without changing the unit of length, and thus changing the *numerical measure* of the accelerating effects of the unit of mass at the *unit of length* in the unit of time.

If the mean sidereal day were taken as the unit of time, the angle n in terms of this unit would become an angle determinate only as a fallible quantity from observation. It would be liable to not only mere chance errors of observation, but to the effects of any neglected long inequalities in the expression of the Sun's longitude: and the use of n^2a^3 for f under such circumstances in our mathematical work would render f a variable quantity instead of a constant. And the errors thus introduced in our mathematical work would be of the same class, but of a more involved character, as those which led to the prolonged discussion on the value of the secular acceleration of the Moon's mean motion.

My contention here is that neither the mean sidereal day nor any assigned definite multiple of that interval of time is adopted as the unit of time in the formation of the differential equations of motion of gravitational astronomy, or that the integrations are erroneously carried out in terms of the variable. Similar remarks apply to the adoption of the physical "mean solar day." The angle n_0 required to fix such a unit of time is at present unknown.

The mathematical results obtained by the writer of this review agree with those which I have obtained; and it would appear that the choice lies between astronomers accepting the fact that they have overlooked and neglected the effects of the small terms

$$\frac{n_0-n}{n_0}$$

in finding their time from observation, or assuming the possibility of the investigations of mathematical astronomy being carried on with units of length and of time, which are discontinuous, and of which the laws of discontinuity are unknown.

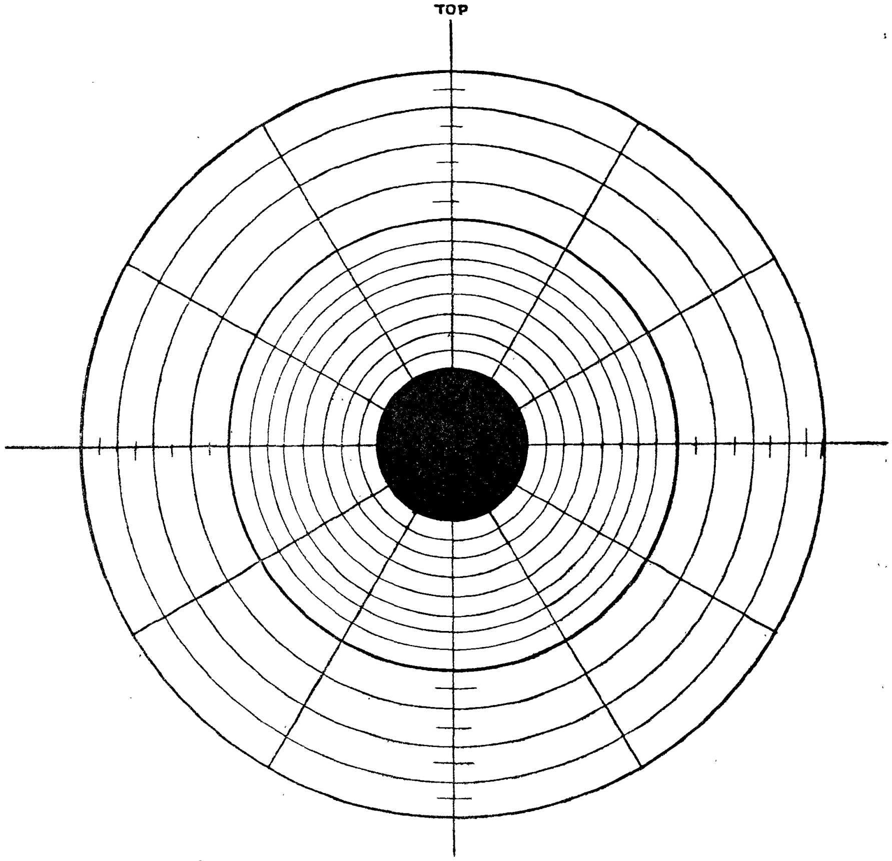
The effects of the errors indicated in finding the variable, t, from observation, when n_0-n is not zero, must be duly allowed for in our work before any inferences respecting the inaccuracy of the law of gravitation or sensible instability in the rotation of the Earth about its axis can be discussed with advantage.

Solar Eclipse without Instrumental Means. By E. J. Stone, M.A., F.R.S., Radcliffe Observer.

A considerable number of persons keenly interested in astronomy, but unprovided with instrumental means, will, if the weather prove favourable, see the eclipse of 1896 August 9 in Norway.

It has occurred to me that such observers might render a service to astronomy if they were to follow out the plan I recommended, and which was carried out under somewhat similar circumstances for the observations of the eclipse of 1874 in South Africa.

The corona consists of a comparatively bright inner part lying close to the Sun, surrounded by a much fainter mass of luminous matter of vast extent, and generally of most irregular form, which does not yet appear to have been successfully photographed to its full extent. Accurate drawings of the outline will be exceedingly valuable, and, fortunately, inaccuracy, such as affects



the scientific value of the drawings, can be avoided if the following precautions are taken: Those persons who intend to make sketches should provide themselves with a sheet of paper about 9 inches wide by 12 inches long, having upon it a black disc